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On Multivariate Approximation by Bernstein-Type Polynomials

MARIE-MADELEINE DERRIENNIC

Laboratoire d'Analyse Numérique, Institut National des Sciences Appliquées, 20, avenue des Buttes de Coësmes, 35043 Rennes Cedex, France

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INTRODUCTION

We study here Bernstein-type polynomial operators defined for integrable functions on a simplex T in $\mathbb{R}^l$. They are generalizations of the modified Bernstein polynomial operators on $L^p(0, 1)$ introduced by J. L. Durrmeyer in [8] and studied by the author in [5]. We denote by $(M_n)_{n \geqslant 1}$ the sequence of those modified operators.

First, we study properties associated with the self-adjointness of M_n and express $M_n f$, for f integrable on T, as a very simple Fourier-type sum. Then, we verify the convergence of derivatives of $M_n f$ to derivatives of f, a known property of the classical Bernstein polynomials, and we find the same rapidity of convergence involving $n^{-1/2}$. Finally, we prove the convergence of $M_n f$ to f belonging to $L^p(T)$, in $L^p(T)$, for $p \geqslant 1$ and estimate the degree of approximation of f by $M_n f$ in $L^p(T)$. (The statements will be given in the general case of a simplex of $\mathbb{R}^l$, but, to simplify, technical proofs will be done in the case $l = 2$.)

I. DEFINITION AND FIRST PROPERTIES

DEFINITION. Let the simplex in $\mathbb{R}^l$ be

$$T = \left\{ X = (x_1, x_2, ..., x_l) \mid x_i \geqslant 0,\ i = 1, 2, ..., l;\ \sum_{i=1}^{l} x_i \leqslant 1 \right\}.$$

The modified Bernstein polynomial of degree n, for a T-integrable function f, is defined by

$$M_n f(X) = \int_T K_n(X, U) f(U)\, dU \qquad \text{for any } X \in T,$$

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where

$$K_n(X, U) = \frac{(n+l)!}{n!} \sum_{|h| \leqslant n} p_{nh}(X) p_{nh}(U)$$

and

$$p_{nh}(X) = \frac{n!}{h!(n-|h|)!} X^h (1-|X|)^{n-|h|}$$

for any h such as $|h| \leqslant n$ (as usual, for $h = (h_1, h_2, ..., h_l)$ in $\mathbb{N}^l$ and $X = (x_1, x_2, ..., x_l)$ in $\mathbb{R}^l$, we denote

$$|h| = \sum_{i=1}^{l} h_i, \qquad h! = h_1!\, h_2! ... h_l!, \qquad |X| = \sum_{i=1}^{l} x_i, \qquad X^h = x_1^{h_1} x_2^{h_2} \cdots x_l^{h_l}).$$

PROPOSITION I.1. *The operator M_n has the following properties of classical Bernstein polynomial operator*:

(1) *It is linear, positive.*

(2) *It preserves the constants, transforms a function $f(x_1, x_2, ..., x_l)$ dependent only on x_k, in a function $M_n f$ dependent only on x_k, for $k = 1, 2, ..., l$.*

(3) *It preserves the degree of polynomials with regard to each variable when their global degree is $\leqslant n$.*

Moreover, the operator M_n is self-adjoint: the equality

$$\int_T M_n f(X) g(X)\, dX = \int_T f(X)\, M_n g(X)\, dX$$

holds for any f and g in $L^1(T)$. (We recall $B_n f(X) = \sum_{|h| \leqslant n} p_{nh}(X)\, f(h/n)$, for $X \in T$.)

Proof. The property (1) comes from the positivity of the kernel K_n. The second one is an immediate consequence of the binomial formula. For the third one, we use Leibniz' formula for the derivative $D^q(X^q(|X|+z)^n)$ at the point $z = 1 - |X|$ (D^q is the differential operator

$$\left.\frac{\partial^{|q|}}{\partial x_1^{q_1}\, \partial x_2^{q_2} \cdots \partial x_l^{q_l}}\right)$$

to obtain the expression of $M_n f$ when $f(X) = X^q$:

$$M_n f(X) = \frac{(n+l)!}{(n+|q|+l)!} \sum_{s=0}^{q} \frac{q!}{s!} \binom{q}{s} \frac{n!}{(n-|s|)!} X^s, \tag{1.1}$$

where, for the multi-integers $q=(q_1, q_2,..., q_l)$ and $s=(s_1, s_2,..., s_l)$, instead of $\sum_{s_1=0}^{q_1} \sum_{s_2=0}^{q_2} \cdots \sum_{s_l=0}^{q_l}$, we denote $\sum_{s=0}^{q}$, and $\binom{q}{s}=q!/s!(q-s)!$.

II. Self-Adjoint Properties

In this part, we study the properties of the operator M_n, fastened to its self-adjointness. We denote $\mathscr{P}_m$, the space of polynomials of global degree $\leqslant m$, and $\mathscr{Q}_m$, the subspace of $\mathscr{P}_m$, orthogonal, for the inner product on $L^2(T)$, to the space $\mathscr{P}_{m-1}$.

THEOREM II.1. *For every $m \geqslant 1$, the space $\mathscr{Q}_m$ is an eigensubspace of M_n associated to the eigenvalue*

$$\lambda_{n,m}=\frac{(n+l)!\, n!}{(n+m+l)!\, (n-m)!}$$

if $m \leqslant n$ and $\lambda_{n,m}=0$ if $m>n$.

Consequently, for any integrable function f, $M_n f$ can be written

$$M_n f=\sum_{m=0}^{n} \lambda_{n,m} P_m f, \tag{2.1}$$

where $P_m f$ is the "projection" of f on the space $\mathscr{Q}_m$ (the inner product, on $L^2(T)$, $\langle f, Q\rangle$, is continuously defined for any integrable function f and any polynomial Q).

Proof. We use a property given in [5] for a Hilbert space H, a subspace J of H, and a linear operator L on H, such as $L(J) \subset J$ and $L^*(J) \subset J$: if a vector V verifies $V \notin J$, $V \perp J$, L and L^* are stable on the direct sum $J \oplus \{V\}$, then V is an eigenvector of L.

For every multi-integer q, with $|q|=m$, we define the polynomial $V_q=X^q+W_q$ such that V_q is orthogonal to $\mathscr{P}_{m-1}$ and W_q belongs to $\mathscr{P}_{m-1}$. We then use the above-mentioned property for $H=L^2(T)$, $L=M_n$, $J=\mathscr{P}_{m-1}$, $V=V_q$. The eigenvalue associated to the eigenvector V_q is obtained as the coefficient of X^q in the polynomial $M_n(X^q)$ and it is

$$\frac{(n+l)!}{(n+m+l)!} \frac{n!}{(n-m)!} \qquad \text{if } m \leqslant n, \text{ zero if } m>n.$$

As this eigenvalue depends only on $|q|=m$, and as the polynomials V_q when q runs on the multi-integers such as $|q|=m$ span $\mathscr{Q}_m$, this space is an eigensubspace of M_n. Let us denote $\lambda_{n,m}$ the eigenvalue of M_n associated to $\mathcal{Q}_m$ and take $Q_{k,m}$, $k=1,2,\dots,m_l$, an orthonormal system spanning $\mathcal{Q}_m$. Since the global degree of $M_n f$ is not greater than n, we have, for any integrable function f, a decomposition

$$M_n f = \sum_{m=0}^{n} \sum_{k=0}^{m_l} \mu_{k,m}(f) Q_{k,m}.$$

Writing $\langle M_n f, Q_{k,m} \rangle = \lambda_{n,m} \langle f, Q_{k,m} \rangle = \mu_{k,m}(f)$, we conclude that (2.1) holds for any f. (m_l is the number of multi-integers q such as $|q| = m$.)

III. Convergence of Partial Derivatives

We prove in this part the convergence of partial derivatives of $M_n f$ to partial derivatives of f (when they exist), a well-known property for classical Bernstein polynomials (cf. for $[0,1]^2$, P. L. Butzer [3]).

THEOREM III.1. *If the function f has a continuous partial derivative $D_q f$ on T, then:*

$$\text{(i)} \quad \sup_{X \in T} |D^q M_n f(X)| \leqslant \sup_{X \in T} |D^q f(X)|, \tag{3.1}$$

$$\text{(ii)} \quad \sup_{X \in T} |D^q M_n f(X) - D^q f(X)| \leqslant C_1 \omega(D^q f, n^{-1/2}) + C_2 n^{-1} \sup_{X \in T} |D^q f(X)|, \tag{3.2}$$

where $\omega(g,\delta)$ is the value of the modulus of continuity of the continuous function g in $\delta > 0$, and C_1, C_2 are two constants dependent only on q and l.

Proof. (in the case $l=2$). If f is integrable on T and $n \geqslant |q|$, we have the first expression for $X \in T$:

$$D^q M_n f(X) = a_{n,q} \sum_{|h+q| \leqslant n} p_{n-|q|,h}(X) \int_T (-1)^{|q|} D^q(p_{n+|q|,h+q}(U)) f(U)\, dU, \tag{3.3}$$

where to simplify the writing we denote

$$a_{n,q} = \frac{n!\,(n+2)!}{(n+|q|)!\,(n-|q|)!}.$$

We obtain this expression, reasoning by recurrence on q_1 and q_2 and using the relation for $h_1 \geqslant 1$, $|h| \leqslant n-1$, $X \in T$:

$$\frac{\partial}{\partial x_1} p_{n,h_1,h_2}(X) = n(p_{n-1,h_1-1,h_2}(X) - p_{n-1,h_1,h_2}(X)),$$

and the similar one about the variable x_2.

Then if f owns a continuous partial derivative $D^q f$ on T, we use Green's formula in (3.3); there appear curvilinear integrals which are zero, so we have the second expression for any $X \in T$:

$$D^q M_n f(X) = a_{n,q} \sum_{|h+q| \leqslant n} p_{n-|q|,h}(X) \int_T p_{n+|q|,h+q}(U)\, D^q f(U)\, dU. \tag{3.4}$$

To prove (3.1), we use then the binomial formula to get, for any $X \in T$, the identity:

$$\sum_{|h+q| \leqslant n} p_{n-|q|,h}(X) = 1 \tag{3.5}$$

and the inequality $a_{n,q}(n+|q|+2)^{-1}(n+|q|+1)^{-1} \leqslant 1$ for any n, q such as $|q| \leqslant n$.

To prove (3.2), we write for any $X \in T$:

$$\begin{aligned} &|D^q M_n f(X) - D^q f(X)| \\ &\quad \leqslant |1 - (n+|q|+2)(n+|q|+1)\, a_{n,q}^{-1}|\, |D^q M_n f(X)| \\ &\quad + |(n+|q|+2)(n+|q|+1)\, a_{n,q}^{-1}\, D^q M_n f(X) - D^q f(X)|. \end{aligned} \tag{3.6}$$

As $|D^q M_n f(X)$ does not exceed $\sup_{U \in T} |D^q f(U)|$, and as there exists a constant C_2, dependent only on q such as $|1-(n+|q|+2)(n+|q|+1)a_{n,q}^{-1}| \leqslant C_2/n$, we have only to consider the last term of the inequality (3.6).

In a well-known way, with the help of the modulus of continuity (cf. C. Coatmelec [4]) and using the relation (3.5), we get for any $X \in T$:

$$\begin{aligned} &|(n+|q|+1)(n+|q|+2)\, a_{n,q}^{-1}\, D^q M_n f(X) - D^q f(X)| \\ &\quad \leqslant (n+|q|+1)(n+|q|+2) \sum_{|h+q| \leqslant n} p_{n-|q|,h}(X) \\ &\qquad \times \int_T p_{n+|q|,h+q}(U)\, |D^q f(U) - D^q f(X)|\, dU \\ &\quad \leqslant \omega(D^q f, \delta)\, [1 + \delta^{-1}(n+|q|+1)(n+|q|+2) \\ &\qquad \times \sum_{|h+q| \leqslant n} p_{n-|q|,h}(X) \int_T p_{n+|q|,h+q}(U) \| U - X \|\, dU], \end{aligned}$$

where $\|\cdots\|$ is the euclidean norm on $\mathbb{R}^2$ and the positive real number δ is to be precised later.

We compute for any $X=(x_1, x_2)\in T$ and $i=1, 2$:

$$\sum_{|h+q|\leqslant n} p_{n-|q|,h}(X)\int_T p_{n+|q|,h+q}(U)\,(u_i-x_i)^2\,dU$$
$$=\frac{(n+|q|)!}{(n+|q|+4)!}\{2nx_i(1-x_i)+[x_i^2(4|q|^2+16|q|+12)$$
$$-2x_i(|q|(2q_i+3)+4(q_i+1))+(q_i+1)(q_i+2)]\}. \quad (3.7)$$

So, with the help of Cauchy–Schwarz inequality for the sums and for the integrals we have

$$(n+|q|+1)(n+|q|+2)$$
$$\times\sum_{|h+q|\leqslant n} p_{n-|q|,h}(X)\int_T p_{n+|q|,h+q}(X)\,\|U-X\|\,dU\leqslant\left(\frac{n+\gamma_q}{n^2}\right),$$

where γ_q is twice the greatest value, for $x_i\in[0, 1]$, of the terms under bracket in (3.7).

We take now $\delta=n^{-1/2}$ and we obtain that the last term of the inequality (3.6) is bounded by $C_1\omega(D^qf, n^{-1/2})$ where $C_1=2+\gamma_q^{1/2}$.

IV. Convergence in $L^p(T)$

In this part, we prove the convergence of M_nf to f in $L^p(T)$ for any $f\in L^p(T)$, $p\geqslant 1$. We give an estimate of the degree of approximation of f by M_nf with the help of the modulus of smoothness of f defined for $\delta>0$ by

$$\omega_p(f, \delta)=\sup_{|h|\leqslant\delta}\left(\int_{T_h}|f(X+h)-f(X)|^p\,dX\right)^{1/p},$$

where $T_h=\{X\mid (X, X+h)\in T\times T\}$.

First, we deal with the problem on the space $C^1(T)$ of the continuous functions with continuous first derivatives.

Then the Peetre-$\mathscr{K}$-functional of the function $f\in L^p(T)$ defined by

$$\mathscr{K}_p(t,f)=\inf_{g\in C^1(T)}\left\{\|f-g\|_{L_p(T)}+t\sum_{|q|=1}\|D^qg\|_{L_p(T)}\right\}$$

will allow us to enlarge the result to the whole $L^p(T)$, for $1\leqslant p<\infty$, and to continuous functions for $p=\infty$, since the functional $\mathscr{K}_p(t,f)$ is "equivalent" to $\omega_p(f, t)$.

Proposition IV.1. *The operator M_n is a contraction on $L^p(T)$ for $p\geqslant 1$.*

Proof. For $p=\infty$, the result comes from constant preserving property of M_n and for $p=1$, in addition, through the self-adjointness of M_n. The general case is then established with the Riesz convexity theorem (cf. N. Dunford and J. T. Schwartz [7]).

THEOREM IV.1. *For any p, $1 \leqslant p \leqslant \infty$, and for any $f \in C^1(T)$, we have the estimate $\| M_n f - f \|_{L^p(T)} \leqslant C_p n^{-1/2} \sum_{|q|=1} \| D^q f \|_{L^p(T)}$, where C_p is a constant dependent only on p and l.*

Proof. (in the case $l=2$). Since M_n preserves the constants, we have for any $f \in C^1(T)$ and any $X \in T$:

$$|M_n f(X) - f(X)| \leqslant \int_T K_n(X, U)\, |f(U) - f(X)|\, dU. \tag{4.1}$$

We begin by the case $p=\infty$.

For any X and U in the simplex T, we get

$$|f(U) - f(X)| \leqslant \left(\left\| \frac{\partial f}{\partial x_1} \right\|_{L^\infty(T)} + \left\| \frac{\partial f}{\partial x_2} \right\|_{L^\infty(T)} \right) \| X - U \|. \tag{4.2}$$

Using the computation (3.7) for $q_1 = q_2 = 0$, we have

$$\sup_{X \in T} \int_T K_n(X, U)\, \| X - U \|^2 \leqslant n^{-1}. \tag{4.3}$$

Summing up the inequalities (4.1), (4.2), and (4.3), after using Cauchy Schwarz inequality we obtain

$$\| M_n f - f \|_{L^\infty(T)} \leqslant n^{-1/2} \left(\left\| \frac{\partial f}{\partial x_1} \right\|_{L^\infty(T)} + \left\| \frac{\partial f}{\partial x_2} \right\|_{L^\infty(T)} \right).$$

We deal now with the case $1 < p < \infty$.

Splitting the set of integration in order to stay in T, we use Hölder inequality, symmetricity of K_n, and (4.1) to obtain

$$\| M_n f - f \|_{L^p(T)} \leqslant 2 \left(\iint_{T \times T} K_n(X, U)\, |f(u_1, u_2) - f(x_1, u_2)|^p\, dU\, dX \right)^{1/p}$$
$$+ 2 \left(\iint_{T \times T} K_n(X, U)\, |f(x_1, u_2) - f(x_1, x_2)|^p\, dU\, dX \right)^{1/p}. \tag{4.4}$$

Introducing the function on $[0,1]^3$, $\phi(u_1, s, x_1) = 1$ if $u_1 \leqslant s \leqslant x_1$ or $x_1 \leqslant s \leqslant u_1$, $\phi(u_1, s, x_1) = 0$ elsewhere, we write, for any (u_1, u_2, x_1) with $(u_1, u_2) \in T$, $(x_1, u_2) \in T$:

$$|f(u_1, u_2) - f(x_1, u_2)| \leqslant |u_1 - x_1|^{1 - 1/p} \left(\int_0^1 \left| \frac{\partial f}{\partial x_1}(s, u_2) \right|^p \phi(u_1, s, x_1)\, ds \right)^{1/p}.$$

So, we obtain

$$\iint_{T\times T} K_n(X, U)\,|f(u_1, u_2)-f(x_1, u_2)|^p\,dU\,dX$$
$$\leqslant \left(\int_T \left|\frac{\partial f}{\partial x_1}(s, u_2)\right|^p ds\,du_2\right)$$
$$\times \sup_{(s,u_2)\in T} \iint_{T\times[0,1-u_2]} K_n(X, U)\,|u_1-x_1|^{p-1}\,\phi(u_1, s, x_1)\,dX\,du_1. \tag{4.5}$$

Let δ be a positive real number which will be precised later ($\delta<1$). For any $X\in T$ and $(s, u_2)\in T$, we split the integral $\int_0^{1-u_2}|u_1-x_1|^{p-1}$ $\phi(u_1, s, x_1)\,du_1$ in two integrals according to $|u_1-s|<\delta$ or not, to bound it by

$$\int_{0\,|u_1-s|<\delta}^{1-u_2} |u_1-x_1|^{r+1}\,|u_1-s|^{p-r-2}\,\phi(u_1, s, x_1)\,du_1$$
$$+\int_{0\,|u_1-s|\geqslant\delta}^{1-u_2} |u_1-x_1|^{r+2}\,|u_1-s|^{p-r-3}\,\phi(u_1, s, x_1)\,du_1, \tag{4.6}$$

where the integer r is defined by $r\leqslant p-1<r+1$.

Hence, for any $(s, u_2)\in T$ and $X\in T$ we have

$$\iint_{T\times[0,1-u_2]} K_n(X, U)\,|u_1-x_1|^{p-1}\,\phi(u_1, s, x_1)\,dX\,du_1$$
$$\leqslant \left(\int_{0\,|u_1-s|<\delta}^{1-u_2} |u_1-s|^{p-r-2}\,\phi(u_1, s, x_1)\,du_1\right)$$
$$\times \sup_{u_1\in[0,1-u_2]} \int_T K(X, U)\,|u_1-x_1|^{r+1}\,dX$$
$$+\left(\int_{|u_1-s|\geqslant\delta} |u_1-s|^{p-r-3}\,\phi(u_1, s, x_1)\,du_1\right)$$
$$\times \sup_{u_1\in[0,1-u_2]} \int_T K(X, U)\,|u_1-x_1|^{r+2}\,dX$$
$$\leqslant \delta^{p-r-1}(p-r-1)^{-1}\xi_n(r+1)+\delta^{p-r-2}(p-r-2)^{-1}\xi_n(r+2), \tag{4.7}$$

where $\xi_n(r)=\sup_{U\in T}\int_T K_n(X, U)\,\|U-X\|^r\,dX$.

We need now a result of the next Proposition IV.2 there exists a constant $C(r)$, independent of n, such as $\xi_n(r)\leqslant C(r)n^{-r/2}$.

So, the right side of (4.5) is bounded by

$$\left\| \frac{\partial f}{\partial x_1} \right\|^p_{L^p(T)} \delta^{p-r-1} n^{-(r+1)/2} C'_p (1+\delta^{-1} n^{-1/2}),$$

where C'_p is a constant dependent only on p.

We choose $\delta = n^{-1/2}$ and we get

$$\iint_{T\times T} K_n(X, U)\, |f(u_1, u_2) - f(x_1, u_2)|^p \, dU \, dX \leqslant 2C'_p \left\| \frac{\partial f}{\partial x_1} \right\|^p_{L^p(T)} n^{-p/2}. \quad (4.8)$$

In the same way, we obtain

$$\iint_{T\times T} K_n(X, U)\, |f(x_1, u_2) - f(x_1, x_2)|^p \, dX \, dU \leqslant 2C'_p \left\| \frac{\partial f}{\partial x_2} \right\|^p_{L^p(T)} n^{-p/2}. \quad (4.9)$$

Summing up the inequalities (4.4), (4.8), and (4.9), it comes

$$\| M_n f - f \|_{L^p(T)} \leqslant C_p n^{-1/2} \left(\left\| \frac{\partial f}{\partial x_1} \right\|_{L^p(T)} + \left\| \frac{\partial f}{\partial x_2} \right\|_{L^p(T)} \right).$$

For the proof in the case $p = 1$, we proceed in the same manner.

The inequalities (4.4) and (4.5) are still true and the integral $\int_0^{1-u_2} \phi(u_1, s, x_1)\, du_1$ is bounded by

$$\int_{|u_1 - s| < \delta} \phi(u_1, s, x_1)\, du_1 + \int_{0, |u_1 - s| > \delta}^{1-x_2} (u_1 - x_1)^2 (u_1 - s)^{-2} \phi(u_1, s, x_1)\, du_1. \quad (4.6)'$$

Then, we continue as for $p > 1$, and the term of the right side in (4.5) is less than

$$2 \left\| \frac{\partial f}{\partial x_1} \right\|_{L^1(T)} (\delta + \delta^{-1} n^{-1}) \leqslant 4n^{-1/2} \left\| \frac{\partial f}{\partial x_1} \right\|_{L^1(T)}. \quad (4.7)'$$

Then we conclude as above for $p > 1$.

Remark. Using an argument of convexity, we could get the property for $1 < p < \infty$ as a consequence of the property for $p = 1$ and $p = \infty$.

PROPOSITION IV.2. *For any integer r, we have the estimate*

$$\sup_{X \in T} \int_T K_n(X, U)\, \| X - U \|^r \, dU = O(n^{-r/2}).$$

Proof ($l=2$). We denote, for $i=1, 2$ and an integer r,

$$S_{n,r,i}(X)=\int_T K_n(X, U)\,(x_i-u_i)^r\,dU.$$

We prove, in the same way as in [5, p. 328], computing $(\partial/\partial x_i)S_{n,r,i}(X)$, the identity

$$(r+n+3)S_{n,r+1,i}(X)=x_i(1-x_i)\left[2rS_{n,r-1,i}(X)-\frac{\partial}{\partial x_i}S_{n,r,i}(X)\right]$$
$$-((1-2x_i)(r+1)-x_i)S_{n,r,i}(X).$$

Reasoning by recurrence on r, we verify that $S_{n,r,i}(X)$, which is a polynomial in x_i of degree r, is a rational fraction in n of degree $-r/2$ if r is even, of degree $-(r+1)/2$ if r is odd. Then, the result follows with the help of Cauchy–Schwarz inequality.

Remark. Theorem IV.1 is still true if f belongs to Sobolev space $W_{1,p}(T)$. Indeed, $C^1(T)$ is dense in $W_{1,p}(T)$ and M_n is a contraction on $L^p(T)$. (the definition of $W_{d,p}(T)$ is recalled next.)

THEOREM IV.2. *Let there be $1\leqslant p<\infty$, for any $f\in L^p(T)$, the sequence $M_n f$ converges to f in $L^p(T)$, and*

$$\|M_n f-f\|_{L^p(T)}\leqslant C_p\omega_p(f, n^{-1/2}).$$

Proof ($l=2$). Let there be f, a function belonging to $L^p(T)$, $p\geqslant 1$. For any g in $C^1(T)$, since M_n is a contraction in $L^p(T)$, we write

$$\|M_n f-f\|_{L^p(T)}\leqslant \|M_n g-g\|_{L^p(T)}+2\|f-g\|_{L^p(T)}.$$

Using Theorem IV.1, this quantity is bounded by

$$C_p n^{-1/2}\sum_{|q|=1}\|D^q g\|_{L^p(T)}+2\|f-g\|_{L^p(T)},$$

which is not greater than $(2+C_p)\mathcal{K}_p(n^{-1/2}, f)$ where $\mathcal{K}_p$ is Peetre-$\mathcal{K}$-functional. Now, we use the well-known result for a Lipschitz-graph domain (see H. Johnen and K. Cherer [9]):

$$\mathcal{K}_p(t, f)\leqslant Cte\,\omega_p(f, t).$$

So we obtain our main desired result.

Remark. The last result is natural and expected; indeed, the degree of approximation corresponding to Bernstein–Kantorovic polynomials of degree n in $L^p(0, 1)$ is indicated by H. Berens and R. A. DeVore in [2]. It is $\omega_p(f, n^{-1/2})$ for any function f in $L^p(0,1)$.

Now let us introduce $W_{d,p}(T)$, Sobolev spaces of functions $f \in L^p(T)$ with derivatives $D^q f$ (in the distributional sense) belonging to $L^p(T)$, $|q| \leqslant d$, endowed with the norm

$$\|f\|_{d,p} = \left(\sum_{|q| \leqslant d} \|D^q f\|^p_{L^p(T)} \right)^{1/p}.$$

PROPOSITION IV.3. *The two assumptions are equivalent for* $p > 1$:

(1) f *is in* $W_{d,p}(T)$,

(2) $\|M_n f\|_{d,p}$ *is uniformly bounded.*

Proof. $(l=2)$. First, let us suppose (1) is true.

Using the density of the space $\mathscr{D}(T)$ of infinitely differentiable functions on T with compact support, in $W_{d,p}(T)$, we show that the expression of $D^q M_n f$, (3.4), falls for any f in $W_{d,p}(T)$ if $|q| \leqslant d$ and $|q| \leqslant n$.

Then Hölder inequality and binomial formula lead us to

$$\|D^q M_n f\|_{L^p(T)} \leqslant \frac{n!\,(n+2)!}{(n+|q|)!\,(n-|q|+2)!} \|D^q f\|_{L^p(T)} \leqslant 2\,\|D^q f\|_{L^p(T)}.$$

So, $\|M_n f\|_{d,p} \leqslant 2\,\|f\|_{d,p}$. This inequality is still true if $p = 1$.

Conversely, let us suppose $\|M_n f\|_{d,p}$ is uniformly bounded. Since $M_n f$ converges to f in $L^1(T)$, for $|q| \leqslant d$, $D^q M_n f$ converges to $D^q f$ in the distributional sense. Then, $\mathscr{D}(T)$ being dense in $L^q(T)$, $1/p + 1/q = 1$, and $\|M_n f\|_{d,p}$ being bounded, $D^q M_n f$ is a weak Cauchy sequence in $L^p(T)$. So its limit is in $L^p(T)$.

COROLLARY. *For any f in $W_{d,p}(T)$, $p \geqslant 1$, the sequence $M_n f$ converges to f in $W_{d,p}(T)$.*

Proof. It is a consequence of Theorem III.1 and of the first implication of Proposition IV.3 for $p \geqslant 1$, via the density of $\mathscr{D}(T)$ in $W_{d,p}(T)$.

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